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Analyzing students' difficulties in understanding real numbers

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Analyzing students' difficulties in understanding real numbers

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Abstract

This article reports on a study of high-school and of technologist students (prospective engineers and economists) understanding of real numbers. Our study was based on written response to a properly designed questionnaire and on interviews taken from students. The quantitative results of our experiment showed an almost complete failure of the technologist students to deal with processes connected to geometric constructions of incommensurable magnitudes. The results of our experiment suggest that the ability to transfer in comfort among several representations of real numbers helps students in obtaining a better understanding of them. A theoretical explanation about this is obtained through the adoption of the conceptual framework of dimensions of knowledge, introduced by Tirosh et al. (1998) for studying the comprehension of rational numbers. Following in part the idea of generic decomposition of the APOS analysis (Weller et al. 2009) we suggest a possible order for development of understanding the real numbers by students when teaching them at school. Some questions open to further research are also mentioned at the end of the paper.

Keywords: real, rational, irrational, algebraic and transcendental numbers, fractions, decimals, representations of real numbers.

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Analizando las Dificultades de los Estudiantes con la Comprensión de los Números Reales

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Resumen

Este artículo presenta un estudio realizado a estudiantes de un instituto y de una escuela técnica superior (ingenieros y economistas) sobre su comprensión de los números reales. Nuestro estudio se basó en las respuestas escritas a un cuestionario diseñado cuidadosamente, y a entrevistas realizadas a los estudiantes. Los resultados cuantitativos de nuestro experimento muestran un fracaso prácticamente absoluto de los estudiantes de ingeniería para lidiar con procesos que conecten construcciones geométricas y magnitudes inconmensurables. Los resultados de nuestro experimento sugieren que la habilidad para transmitir una cierta soltura en el uso de ciertas representaciones de los números reales ayuda a los estudiantes a obtener una mejor comprensión de los mismos. Una explicación teórica de esto se obtiene a partir del uso del marco conceptual de las dimensiones del conocimiento, introducido por Tirosh et al. (1998) para estudiar la comprensión de los números racionales. Siguiendo en parte la idea de la descomposición genérica del enfoque APOS (Weller et al., 2009), sugerimos un posible orden para el desarrollo de la comprensión de los números reales de estudiantes cuando se les enseña este tema en la escuela. Al final se mencionan algunas preguntas que quedan abiertas para futuras investigaciones.

Palabras Clave: números reales, racionales, irracionales, algebráicos y transcendentes, fracciones, decimales, representación de números reales.

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The empiric approach of numbers starts from pre-school age, when children distinguish the one among many similar objects and count them (Gelman, 2003). This first acquaintance with numbers helps significantly in understanding the structure of the set N of natural numbers. For example, it contributes in clarifying the principle of the "next natural number" leading to the conclusion that N is an infinite set (Hartnett & Gelman, 1998). Further it supports the development of strategies for addition and abstraction that are based on counting (Smith et al., 2005), the comparison between natural numbers (definition of order in N), etc. All these are strengthened during the first two years of primary school, where pupils are usually studying the natural numbers and their operations up to 1000.

Fractions and decimals are also introduced in primary school at a later stage, while negative numbers are usually introduced at the first class of the lower secondary education (i.e. at the 7th grade). It is well known that students face significant difficulties in understanding rational numbers (e.g. Smith et al., 2005). Many of these difficulties are due to the improper transfer of properties of natural numbers to rational numbers (Yujing & Yong-Di, 2005; Vamvakousi & Vosniadou, 2004, 2007). For example, many students believe that "the more elements a number has, the bigger it is" (Moscal & Magone 2000), or that "multiplication increases, while division decreases numbers" (Fischbein et al. 1985). They also believe that the principle of "next number" holds for rational numbers as well (Malara, 2001; Merenluoto & Lehtinen, 2002).

Another characteristic of rational numbers that possibly affects negatively their understanding is their multiple representations (e.g. we can write 1/2 = 2/4 = ... = 0.5). In fact, novices tend to categorize objects in terms of their surface rather, than their structural characteristics (Chi et al., 1981), therefore they face difficulties in understanding that different symbols may represent the same object (Markovitz & Sowder, 1991). Consequently many of them think that different representations of a rational number represent different numbers (Khoury & Zazkis, 1994; O'Connor, 2001) and even more that decimals and fractions are two disjoint subsets of the set Q of rational numbers. We notice that the above false conception is taken roots as a habit even to many adults, who consider in all cases fractions as parts of the whole (e.g. 1/4 of something), while there also exist other considerations for fractions, e.g. as a ratio, as an operator, as the accurate quotient of a division, etc. On the contrary, they consider decimals only as quotients of divisions (e.g. 1÷4=0.25) that have a resemblance with natural numbers.

An essential pre-assumption for the comprehension of irrational numbers is that students have already consolidated their knowledge about rational numbers and, if this has not been achieved, as it usually happens, many problems are created. It has been observed that pupils, but also university students at all levels, are not able to define correctly the concepts of rational and irrational numbers, neither are in position to distinguish between integers and these numbers (Hart, 1988; Fischbein, et al., 1995). It seems that the concept of rational numbers in general remains isolated from the wider class of real numbers (Moseley, 2005; Toepliz, 2007).

Several reports document students' difficulties on the topic of repeating decimals, particularly confusion over the relationship between 0.999… and 1 (Tall & Schwarzenberger, 1978; Hewitt, 1984; Hirst 1990; Sierpinska, 1987; Edwards & Ward, 2004; Weller, Arnon, & Dubinsky, 2009, 2011, etc.). Students in the above reports were expected to realize that converting 0.999… to a fraction (or in some other way) one finds that 0.999… = 1. However, mathematically speaking, there exists actually a confusion even among the mathematicians concerning the truth or not of the above equation. (see Appendix 2)

Research focussed on the comprehension and proper didactical approach of irrational numbers is rather slim. It seems however that, apart from the incomplete comprehension of rational numbers, they are also other obstacles (cognitive and epistemological) that make the comprehension of irrational numbers even more difficult (Herscovics, 1989; Sierpinska, 1994; Sirotic & Zazkis, 2007a; etc).

Fischbein et al. (1995) assumed that possible obstacles for the comprehension of irrational numbers could be the intuitive difficulties that revealed themselves in the history of mathematics, i.e. the existence of *incommensurable magnitudes* and the fact that the *power of continuum* of the set *R* of real numbers is higher than the power of *Q,* which, although being an everywhere dense set, can not cover all points of a given interval. Their basic conclusion resulting from their experiments with school students and pre-service teachers is that school mathematics is generally not concerned with the systematic teaching of the hierarchical structure of the various classes of numbers. As an effect, most of high school students and many pre-service teachers are not able to define correctly the concepts of rational, irrational and real numbers, neither to identify various examples of numbers. They also found that, contrary to their initial assumption, the concept of irrational numbers does not encounter in general a particular intuitive difficulty in students' mind. Hence they assumed that such difficulties are not primitive ones and they express a relatively high level of mathematical education. However they suggest that for a better understanding of irrational numbers teacher should turn students' attention on these difficulties rather, than ignore them.

Peled & Hershkovitz (1999) when performing an experimental research observed that pre-service mathematics teachers being at their second and third year of studies, although they knew the definitions and basic characteristics of the irrational numbers, they failed in tasks that required a flexible use of their different representations. Further, Sirotic & Zazkis (2007b) focusing on the ability of prospective secondary teachers in representing irrational numbers as points on a number line observed confusion between irrational numbers and their decimal approximation and overwhelming reliance on the latter. They also used (Zazkis & Sirotic, 2010) the distinction between transparent and opaque representations of concepts (Lesh et al., 1987) as a theoretical perspective in studying the ways in which different decimal representations of real numbers influenced their responses with respect to their possible irrationality.

According to Lesh et al. (1987) a transparent representation has no more and no less meaning than the represented ideas or structures. On the contrary, an opaque representation emphasizes some aspects of the ideas or structures and de-emphasizes others. For a practical approach of transparent and opaque representations of real numbers we give the following examples:

The rational numbers 3/5 = 0.6, 1/3 = 0.33..., 281849/99900 = 2.821131131131... have transparent decimal representations, since one can foresee their decimal digits in all places; but the same is not happening with 144/233 = 0.61802575107..., which, possessing a period of 232 digits, has an opaque decimal representation. The decimal representations of irrational numbers are opaque in most cases due to their complex structure, but there are also irrational numbers having transparent representations. This happens for example with the numbers 2.00131311311131111311113... where 1, following 13, is repeated one more time at each time, and 0.28228822288822228888 2... where 2 and 8, following 28, are repeated one more time at each time. We shall return with more examples on transparent and opaque representations of real numbers and their important role for the understanding of real numbers by students.

The Experimental Research

Janvier (1987) describes the comprehension of a concept in general as a cumulative process mainly based upon the capacity of dealing with an ever- enriching set of representations. In particular, an extended research has been developed on the role of representations for the better understanding of mathematics (Goldin & Janvier, 1998; Goldin, 2008; Godino & Font, 2010, etc.). Reflecting on the results of this general research as well as on findings of experimental researches on real numbers already mentioned above (Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007b) we led to the hypothesis that the main obstacle for the understanding of real numbers is the intuitive difficulties that students have with their multiple *semiotic representations*, i.e. the ways in which we describe and we write them down. In constructing the theoretical framework of our research we put the following targets:

- To check our basic hypothesis that students' difficulties to deal successfully with the multiple semiotic representations of real numbers is the main obstacle towards their understanding.
- To verify the existence of other obstacles mentioned by other researchers, e. g. the incomplete understanding of the rational before studying the irrational numbers, the intuitive difficulties with the perception of incommensurable magnitudes and the "property of the continuum of *R*", etc.
- To investigate if other factors like the age, the breadth of the

mathematical material covered by students, etc, affect the comprehension and the better use of real numbers.

Our basic tool in our experiment was a questionnaire of 15 questions (see Appendix 1) designed with respect to the above targets. In fact, with question 1 we wanted to check if the students were in position to distinguish the category in which a given number belongs. Questions 2-5 were designed to check the degree of understanding of rational umbers by students. Further, questions 6-8 and 13 were designed to check if students were able to deal in comfort with the square roots of positive integers, while questions 9-12 were connected with the density of *Q* and *R*. Finally with question 14 we wanted to investigate students' ability to deal with geometric representations of real numbers and with question 15 we wanted to check if they realize that the set of irrational numbers is not closed under addition.

Notice that the two authors studied carefully analogous questionnaires of similar experiments performed earlier by other researchers (see back in section 1), they had extensive discussions on the choice and suitability of the questions involved and they attempted (together and separately) several pilot experiments in a smaller scale before reaching the final form of the above questionnaire. At ay case, there is no claim that our final was the best possible. For example, it seems that there was a problem with the choice of question 14 (explained at the end of this section) in favor of the students of Gymnasium. Nevertheless, in general lines the questionnaire was proved in practice to be useful in investigating the above mentioned targets of our research.

A printed copy of the above questionnaire (in Greek language) was forwarded to 78 students of the second class of 1st Pilot Gymnasium of Athens (13-14 years old), one of the good public schools of lower secondary education in Greece, by the end of school year 2008-09, i.e. a few months after learning about real numbers. At the same time the above questionnaire was also forwarded to 106 students of Graduate Technological Educational Institute (T.E.I.) of Patras, from two departments of the School of Technological Applications (prospective engineers) and one department of School of Management and Economics, being at their first term of studies (18-19 years old). The students of T.E.I. had of course much more mathematical experiences than the 14 years old students of Gymnasium, but, according to the marks obtained in the exams for entering tertiary education, they are considered to be moderate graduates of secondary education in general. The choice of the subjects of our experiment was not made by chance, neither because we had an easier access to them. Our purpose was to compare the data obtained from two groups of different ages hoping to obtain some conclusions about the possible effects of age and of individual's mathematical background in understanding better and making a correct use of real numbers. As far as we know, a similar experiment was performed in past only by Fischbein et al. (1995), whose study concerned, apart from high school students, prospective teachers of mathematics, who logically must had studied more carefully the system of real numbers that would be in future one of their basic objects of teaching. On the contrary, students of T.E.I. are using mathematics as a tool for studying and better understanding their sciences (prospective engineers and economists).

The time given to students to complete in writing the questionnaire was one hour. The students' answers were characterized as correct (C) and wrong (W). In few cases of incomplete answers the above characterization created some obscurities, which nevertheless didn't affect significantly the general image of student's performance. In Voskoglou & Kosyvas (2011) we reported in detail the percentages (with unit approximation) of the correct and wrong answers given by students for each question, separately for Gymnasium and T.E.I. Therefore here we shall give only two examples of coding in order to be understood how exactly the data of the experiment were analyzed.

The following matrix gives the percentages of wrong answers given by students in question 1:

Table 1

	0W	1-2 W	3-5 W	6-10 W	> 10 W
Gymnasium	0	5	22	21	52
T.E.I.	0	11	33	36	20

The most common mistakes were the identification of the symbol of fraction with rational and the symbol of root with irrational numbers. The failure of many students to recognize that all the given numbers were real numbers was really impressive. Notice that no students gave correct answers for all cases.

Table 2

	3C	2C	1C	3W
Gymnasium	27	20	13	40
T.E.I.	0	1	1	98

With this question we wanted to investigate the students' ability to construct incommensurable magnitudes and to represent irrational numbers on the real axis. The answers of students of T.E.I. were really an unpleasant surprise. Nobody constructed the length √3 correctly, only two of them constructed √2 and only one found the point corresponding to it on the real axis! On the contrary, the high-school students, recently taught the corresponding geometric constructions, had a much better performance.

A similar analysis was attempted for all the other questions. Next and in order to obtain a statistical image of students' performance the completed by them questionnaires were marked in a scale from 0 to 20. A number of units was attached to each question according to its difficulty and the mean time required to be answered (see Appendix). In the diagram of five numbers' summary [maximum (x_{max}) and minimum (x_{min}) graduation, median (M), first (Q_1) and third (Q_3) quarter] of the total sample the median is lying to the left part of the rectangular formed, which indicates accumulation to low marks (Figure 1).

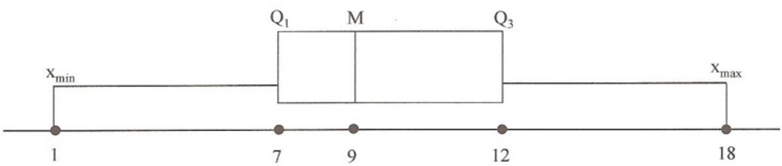

Figure 1. Five number's summary of total sample

The means obtained, 9.41 for Gymnasium, 9.49 for T.E.I. and 9.46 in total, show that students' general performance was insufficient with regard to dexterities and cognitive capacities for real numbers evaluated by the questionnaire. Nevertheless, from both samples becomes evident that students possessed some basic abilities and therefore a great part of the deficiencies observed could be corrected.

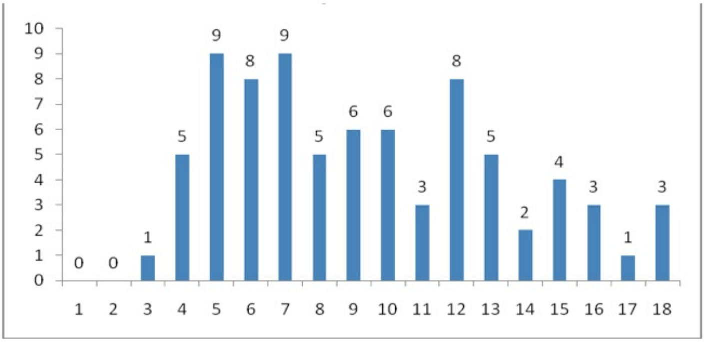

Figure 2. Frequencies of marks for Gymnasium

The diagrams of frequencies of marks separately for Gymnasium (Figure 2) and T.E.I. (Figure 3), where marks are shown on the horizontal axis and the numbers of students obtained the corresponding marks on the vertical axis, give a descriptive view of our experiment's data.

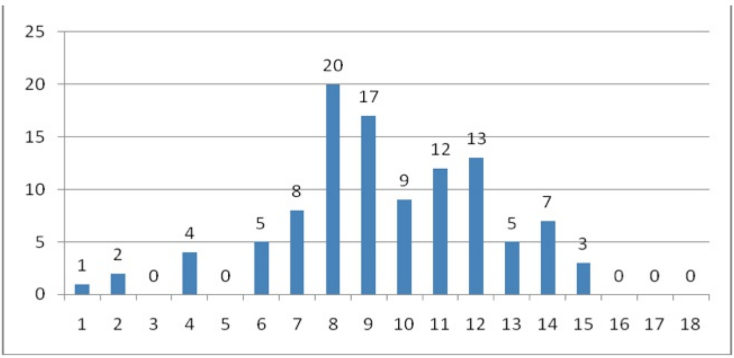

Figure 3. Frequencies of marks for T.E.I.

Finally, the diagram of the percentenges of marks for Gymnasium and T.E.I. together (Figure 4) gives to the reader a better acess in making the necessary comparisons.

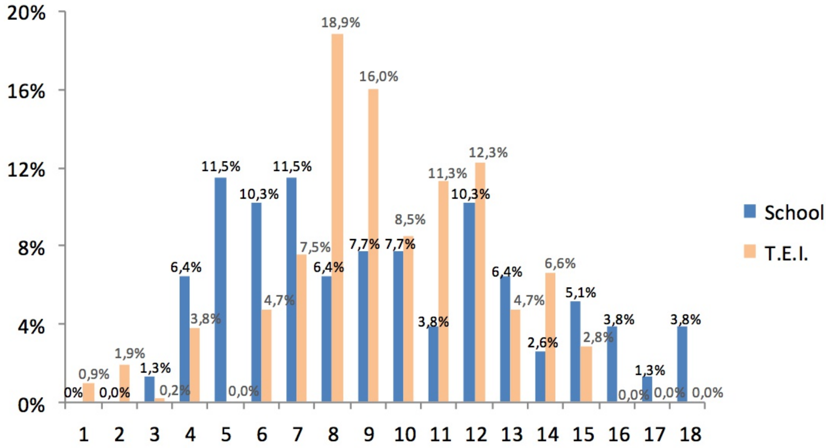

Figure 4. Comparative diagram of percentages of marks of Gymnasium and T.E.I.

The general conclusions obtained through the evaluation of our experiment's data are the following:

- The understanding of rational numbers was proved to be incomplete by many students (questions 1-5 and 9-12). In general students worked in more comfort with decimals rather, than with fractions (questions 11, 12, etc). Further, students who failed to give satisfactory answers to questions 1-5 and 9-12, failed also in answering satisfactorily the rest of the questions. This obviously means that, the incomplete understanding of rational numbers is in fact a great obstacle for the comprehension of irrational numbers.
- Our basic hypothesis that the main obstacle for the understanding of real numbers has to do with students' intuitive difficulties with their multiple semiotic representations was verified in general (questions 5, 8, 13, etc) with a characteristic exception: The students of T.E.I. showed an almost complete weakness to deal with processes connected to geometric constructions of incommensurable magnitudes and to the representation of the irrational numbers on the real axis (question 14). However this didn't prevent them in answering satisfactorily the other questions.
- The density of rational and irrational numbers in a given interval doesn't seem to be embedded properly by a considerable number of

students, especially by those of high-school (questions 9-12).

• It seems that the age and the width of mathematical knowledge affect in a degree the comprehension of the real numbers. In fact, although the majority of the T.E.I. students corresponded to mediocre graduates of secondary education, the superiority of their answers was evident in most of the questions (apart from 3, 7, 8 and 14).

The negligible difference of means of students' marks between Gymnasium and T.E.I. does not represent the real situation (evident superiority of T.E.I. students'). In fact, the means have been formed at this level for two reasons: First, because of the total failure of the T.E.I. students' in answering question 14, which had the highest graduation (2.5 units). Second, because of the high marks (16-18) obtained by a number of students of Gymnasium in contrast to the students of T.E.I. whose marks were below 16.

There are reasonable explanations about these facts: The total failure of T.E.I. students' in constructing geometrically irrational lengths by using the Pythagorean Theorem is probably related to the low attention given today in Greece to the teaching of Euclidean Geometry at the higher level secondary education (Lyceum). On the contrary, the students of Gymnasium, who recently had taught the corresponding geometric constructions, had a better success on this topic. This is an example of the impact that instruction could have on the students' performance.

Also, the fact that a number of students of Gymnasium, which is one of the good (pilot) high schools of Greece, obtained high marks (16-18) in contrast to the students of T.E.I. is not surprising, since the students of T.E.I. correspond to mediocre graduates of the secondary education in general.

A New Qualitative Research

Reflecting on the answers appeared in the completed questionnaires of our experimental research we considered useful to penetrate deeper to the reasons that urged students in giving these answers. Therefore we decided to make a complementary qualitative research by taking some interviews from students. We conducted 20 in total interviews, 10 for the students of T.E.I. by the first author and 10 for high school students by the second author. The choice of students was based on the type of their written answers (answers needing a further clarification) and on their will to participate. The interviews were conducted by appointment in the offices of the researchers and type recorders were used to save them. The two researchers worked together to study and analyze the interviews. Many of the students' answers given during the interviews were similar and therefore they were grouped. We present and analyze below the most representative parts of the interviews separately for the students of Gymnasium and of T.E.I.

Gymnasium

> Question: Why did you answer that - $\sqrt{5}/2$ is a rational number?
> Answer: Because it is a fraction.

In this case we have a classical misinterpretation of the definition of rational numbers. The student focused her attention on the symbol of fraction without realizing that, in order to be a rational number, its terms must be integers, with non-zero denominator.

> Question: Why did you answer that -$\sqrt{4}$ is an irrational number?
> Answer: Because it has the root.

Here the student identifies the symbol of the root with an irrational number. He does not think that the given number is equal to -2, which is an integer. Distinction among several types of numbers remains muddy in general, each time depending on their semiotic representations.

> Question: Which is the exact quotient of the division 5÷7?

The student answered that the exact quotient of the division 5÷7 is 0.714285714285 and that he found it with a calculator. Only when he was asked by the teacher to perform the division by hand he realized that it never ends and that the result is a periodic decimal. In general, the identification of a real number with its given rational approximation (e.g. identification of π with 3.14) is a common mistake in students' responses.

> Question: Why did you answer that 2.00131311311131111131111…
> is a periodic number?
> Answer: Because the decimal digits following 00 are repeated in a

> concrete process: 13, 131, 1311, 13111, 131111, etc. (he explains it orally).
> Question: And does it mean that this is a periodic number?
> Answer: Of course!
> Question: Why is this so?

The student referred to the mathematics text book of his class, where (p. 187) we read that π is not a periodic number, since its infinite decimal digits are not repeated in a concrete process.

Our initial impression was that no student had observed the regularity appearing in the decimal digits of the above number (Voskoglou & Kosyvas, 2011, question 5). The explanation given from the (very good) student for his answer has to do with a superficial definition of irrational numbers. Frequently in text books irrational numbers are defined (correctly) as non rational numbers (they cannot be written as fractions with integer terms), but there is no attempt to identify them with the incommensurable decimals, which are not defined explicitly. Some examples are simply given for the approximate calculation of square roots having no exact values and it is reported that, apart from such roots, there exist other types of real numbers as well, like π.

Consequently, if the teacher does not make the necessary interventions urging students to think on these things, children will probably remain with the doubt: *What is the form of irrational numbers in general?*

> Question: Why did you answer that there is no rational number between fractions 1/10 and 1/11, as well as between decimals 10.20 and 10.21?
> Answer: Because 1/11 is the next fraction of 1/10 and 10,21 is the next decimal of 10.20.

Student's belief in this case is that both fractions and decimals have a next number, which is a classical case of improper transfer of a property of natural numbers to rational numbers. Taking such opportunities, teacher could point out (although this cannot be easily understood without the notion of equivalent sets) that, if in the everywhere dense set of rational numbers we characterized a number as the "next" of another one, we should have omitted as many numbers as the whole set Q has.

Question: Why did you answer that $x=\sqrt{3}$ is the unique root of the equation $x^2=3$?
Answer: Because we know that the square root of 3 is a positive number, such that $x^2=3$.
Question: But $(-\sqrt{3})^2=3$, is n't it?
Answer: You are right, $x=-\sqrt{3}$ is also a root of the above equation.

Although enough hints are contained in student's text book concerning solution of equations of the form $x^2=\alpha$, the restriction imposed that the square root must be a positive number it seems to create some confusion to students. By accepting that for each $x>0$ there exist two square roots, one positive denoted $\sqrt{x}$ and the negative one $-\sqrt{x}$, this confusion could be overcome. Using the previous notation we have no problem in considering the relation $f(x)=\sqrt{x}$ as a function, which is the basic argument of the supporters of definition of square root as a positive number. The rejection of the negative root, although it focuses in keeping the one-valued property of the above function, is a restriction that, among the others, does not permit students in understanding roots as the inverse process of the raising to a power. Furthermore, we believe that it is unnatural to accept, extending the restriction to roots of any order (as it usually happens in the text books of mathematics), that $\sqrt[3]{-8}$ does not exist, despites to the fact that $(-2)^3=-8$. This in a later stage forces us to accept that the domain of the function $f(x)=\sqrt[3]{x}$ is the set of positive numbers, which intuitively cannot be easily accepted. However, for the moment that in school books of mathematics the square root is defined as a non negative number, teacher could be better to give emphasis to the reasons of adopting this definition rather, and not to its mechanical use.

T.E.I.

Question: Which is the exact quotient of the division $5 \div 7$?
Answer: There is no exact quotient, since the division's result is an infinite decimal.
Question: What is the result and how did you find it?
Answer: It is 0.71428571428… and I found it by using a calculator.
Question: Could you carry out the division by hand?
Answer: Yes, but why?

Answer: I will explain in a while.
In fact, student starts performing the division and after 6 steps he finds 5 as remainder. At this point the instructor asks:
Question: Are you observing anything now?
Answer: (Rather perplexed): No.
Question: The last remainder that you found is the same with the initial dividend. What does it mean?
Answer: (After thinking for a while): The same process will be repeated again.
Question: How many times?
Answer: As much as we want (he is thinking…), infinitely many.
Question: Consequently what will happen with the quotient ?
Answer: The decimal digits 714285 will be repeated continuously. Oh! I remember now. We have found a periodic decimal, which is the exact quotient of the division.
Question: Correct. Nevertheless, there is no other way to represent the exact quotient of the division, apart of writing it as a periodic decimal?
Answer: (He is thinking): I don't think so.
Question: What about the fraction 5/7?
Answer: (Surprised): Oh, yes! This is in fact the exact quotient of the division.

An evident difficulty is revealed here in distinguishing between different semiotic representations of rational numbers. Student had not clarified that the exact quotient is the fraction 5/7, or alternatively the periodic decimal 0.714285714285... He simply agreed condescendingly with teacher's view about the first following a question that disclosed the correct answer. We must notice that frequently in text books is not given emphasis to the fact that a fraction represents, among the others, the exact quotient of the division of two integers. On the other hand, student performing the division 5÷7 by a calculator, as it usually happens today, was not helped in recognizing the periodicity of the quotient, since the result obtained happened to be an opaque decimal representation of it. On the contrary, performing the division by hand he had the chance to realize, with the help of the teacher, that from the moment where the same remainder was reappeared, the same process would be repeated infinitely many times and therefore we shall have a continuous appearance of the six digits' period. In other words, students performing the division between integers by hand could be exercised better in recognizing the periodicity of the quotient:

> Question: Why did you consider the equality as $\sqrt{(1-\sqrt{17})^2} = 1-\sqrt{17}$ a correct one?
> Answer: By applying the property $\sqrt{x^2} = |x|$.
> Question: So $\sqrt{(-3)^2} = -3$?
> A. No, square root is always a positive number. (He is thinking…). I am sorry, I made a mistake. The correct is that $\sqrt{x^2}=|x|$.
> Question: That is $|1-\sqrt{17}| = 1-\sqrt{17}$?
> Answer: Yes.
> Answer: But in this case we should have that $1>\sqrt{17}$.
> Answer: Ops! I am sorry. My answer was wrong. The above equality is not correct.

A superficial application of properties of roots is appeared n this case and of the definition of the absolute value that have not been properly assimilated.

> Question: Why did you answer that $(\sqrt{3} + 2)(\sqrt{3} - 2)$ is an irrational number?
> Answer: Because it is a product of two irrational numbers.
> Question: Could you make the multiplication?
> Answer: Of course (she performs the corresponding operations by using the distributive law). Ops, I am wrong! The result is -1.

The student had in this case the wrong belief that multiplication is a closed operation in the set of irrational numbers. Teacher should turn students' attention about this illusion earlier in high school, when they learn the real numbers for first time.

From many of the above dialogues (questions 1-4 for Gymnasium and 1 for T.E.I.) it becomes evident the students' difficulty in dealing successfully with the multiple representations of real numbers (fractions, periodic and non-periodic decimals, roots, etc.). A theoretical explanation about this can be obtained through the conceptual framework of *dimensions of knowledge*, introduced by Tirosh et al. (1998) for studying the comprehension of rational numbers. Their basic assumption is that learners' mathematical knowledge is embedded in a set of connections among the following dimensions (types) of knowledge:

- Algorithmic dimension, concerning individual's ability in applying rules and prescriptions to explain the successive steps involved in various standard operations.
- Formal dimension, concerning the ability of recalling and applying definitions of concepts, theorems and their proofs in problem-solving situations.
- Intuitive dimension, composed of learner's intuitions, ideas and beliefs about mathematical entities and including cognitive models used to represent number concepts and operations. This is the type of knowledge that we tend to accept directly and confidently. It is self-evident and psychologically resistant (Fischbein, 1985).

It seems that people tend to adapt their formal and algorithmic knowledge to accommodate their beliefs (i.e. the conclusions of their intuitive knowledge), perhaps as a natural tendency towards consistency. Therefore, when their beliefs are not clear and/or accurate, it is very possible to lead to mistakes and/or inconsistencies. This is exactly what happens with the multiple representations of real numbers, In fact, as we have already seen above, students are frequently thinking that different representations of the same fraction are different numbers, that fractions and decimals or roots and decimals are sets of numbers disjoint to each other, that infinite decimals are equal to their given finite approximations (e.g. π=3.14, e=2.71, 144/233 = 0.6180257, etc.) and so on. All these wrong beliefs, when they have been formed in the individual's cognitive structures, it is very difficult, according to the explanations provided by the conceptual framework of the dimension of knowledge, to be changed later.

Teaching Real Numbers at School

Weller et al. (2009, 2011) report on the mathematical performance of pre-service elementary and middle school teachers who completed a specially designed experimental unit on repeating decimals that was based on APOS (Action, Process, Object, Schema) theory and implemented using the ACE (Activities in the computer, Classroom discussion, Exercises done outside of class) teaching style. The quantitative results of their experiments suggest that the students who received the experimental instruction made considerable progress in their development of understanding the relation between a rational number (fraction or integer) and its decimal expansion.

The implementation of APOS theory as a framework of learning and teaching mathematics involves a theoretical analysis of the concepts under study in terms of the mental constructions a learner might take in order to develop understanding of the concepts, called a generic decomposition (GD). It comprises a description that includes actions, processes and objects, which describe the order in which it may be best for learners to experience them. While we do not fully employ the idea of a GD here, the construct is a useful one and helped in suggesting a possible order for the development of the understanding of real numbers. The following suggestions were based not only on the outcomes of our experimental research presented above, but also on our many years didactical/pedagogical experience in secondary and tertiary education.

There are several methods known for the construction of the set of real numbers (Voskoglou & Kosyvas 2011, section 3). Apart of their representation as infinite decimals (where a finite decimal can be written as an infinite one, with period equal to 0 or to 9, e. g. 2.5 = 2.500... = 2.499...) the rest of these methods are too abstract to deal with in a regular curriculum for school mathematics.

Two prerequisites seem to be indispensable for a successful presentation of real numbers as infinite decimals at school:

- First, students must have realized that periodic decimals and fractions are the same numbers written in a different way.

• Second, the definition of non periodic decimals must be given in a strict and explicit way, so that it could not give rise to any misinterpretations: An infinite decimal is a non periodic decimal not because its decimal digits are not repeated in a concrete process (this in fact could happen, as the relevant examples given in our introduction show), but because it has not a period, i.e. its decimal digits are not repeated in the same concrete series.

The first of the above prerequisites helps students to realize the equivalence between the two definitions of irrational numbers given at school: As non rational numbers (i.e. they cannot be written as fractions μ/v, $\mu,v \in Z$, $v \neq 0$) and as incommensurable decimals on the other hand. For this, students must have clearly understood that, for each fraction μ/v, μ, $v \in Z$, $v \neq 0$, the quotient of the division $\mu \div v$ is always a periodic decimal. The probability to be a finite decimal is small enough, since a fraction, whose denominator is not a product of powers of 2 and/or 5, cannot be written as a finite decimal. In case of an infinite decimal, students must be in position to observe that, since the remainder of the division $\mu \div v$ is smaller that v, performing the division and after a finite number of steps the same remainder will reappear at some step. This means that from this point and so on the same digits will appear periodically in the quotient again and again, infinitely many times. Conversely, students must be in position to convert periodic numbers (either simple ones, or mixed) to fractions. We recall that a standard method for doing this (although they are others as well) is by subtracting both members of proper equations containing multiples of a power of 10 of the given number. For example, given $x= 2.75323232\ldots$, we write $10000x = 27532.3232\ldots$ and $100x = 275.3232\ldots$, wherefrom we find $9900x = 27532 - 275$, or $x = 27257 / 9900$.

An instructional treatment for the definition of non periodic decimals could be to ask students to calculate the finite approximations of square roots of non quadratic positive integers. For example, $\sqrt{2}$ is written as $\sqrt{2}$ and is constructed as the limit of the sequence 1, 1.4, 1.41, 1.414, 1.4142... of its finite (rational) approximations.

The concepts of a sequence of rational numbers and of its limit (i.e. what it means to "tend" to a number) should be presented in a practical way by teacher (the detailed study of these topics is a didactic object in an upper level of studies) and explained to students through the above examples. The dots at the end of the number indicate that the sequence of its decimal digits is continued. Students must understand that the acceptance of this symbolic representation of an infinite decimal does not mean that we can see written all its decimal digits. We can only see the digits of its given decimal approximation each time.

For students it is difficult in general to understand a number if they don't know an explicit way of writing it down. Therefore it is very important to give frequently opportunities to them to rethink critically about the decimal representations of real numbers. For example, let us consider the following (vertical) pairs of numbers:

$$\begin{cases} \dfrac{22}{7} = 3.14285714... \\ \pi = 3.14159265... \end{cases} \qquad \begin{matrix} \dfrac{144}{233} = 0.6118025... \\ \dfrac{\sqrt{5}-1}{2} = 0.618033... \end{matrix} \qquad \begin{matrix} \dfrac{1}{1861} = 0.0005373.... \\ \dfrac{1}{\sqrt{3499149}} = 0.0005345... \end{matrix}$$

The rational numbers of the first row have a period of 6.232 and 1860 digits respectively, while the irrational numbers of the second row have not any regularity concerning the appearance of their decimal digits. As most of the decimal digits of all the above numbers remain unknown, given only their decimal representations you cannot be sure where they are, or not, rational numbers . In fact, although a number of digits of the above vertical pairs of numbers coincide, the rest of them remain unknown. As a result their possible rationality or not depends upon the completion of their decimal representations with their opaque parts. In converting a fraction to a decimal a long and laborious division is reached, if the quotient obtained is an infinite decimal having a long period, which is not possible to be determined soon.

We also observe that, if we restrict the decimal representations of the above numbers to their digits written in bold only, then they take the following form:

$$\begin{cases} 3.14... \\ 3.14... \end{cases} \qquad \begin{matrix} 0.6180... \\ 0.6180... \end{matrix} \qquad \begin{matrix} 0.00053... \\ 0.00053... \end{matrix}$$

Now the decimal representations of the corresponding vertical pairs of numbers coincide to each other. Consequently it is completely impossible to conclude whether they are rational numbers, or not.

Problems however are increasing when we arrive to the expected (since students already know that fractions can be written as periodic decimals) question: *Which numbers can be written as incommensurable decimals?* Firstly, students realize that this happens with the square roots of non quadratic positive rational numbers. Later they learn that the same happens with the roots of any order whose value is not a finite decimal. Nevertheless they are also irrational numbers having not this form, or, in a more general context, numbers which are not roots of a polynomial equation with coefficients in *Q,* i.e. which are not algebraic numbers. In this way we approach the concept of *transcendental numbers*, with π and e being the better known examples. It can be shown that the set of algebraic numbers is a denumerable set, while Cantor proved that the set of transcendental numbers has the power of continuous. This practically means that transcendental are much more than algebraic numbers, but the information that we have about them is very small related to their multitude. That is why we have characterized them as a "black hole" (with the astronomical meaning of term) in the "universe" of real numbers (Voskoglou, 2011). The instructor could give to students a brief description of algebraic and transcendental numbers, so that to obtain a complete view of the whole spectre of real numbers. A good opportunity is given in reviewing the basic sets of numbers at Lyceum before studying the complex numbers. However references at earlier stages are not excluded, since this new kind of numbers usually activates students' imagination and increases their interest by creating a pedagogical atmosphere of mystery and surprise.

The quantitative results of our experimental research (section 2) show that the complete failure of the students of T.E.I. to deal with processes connected to geometric constructions of incommensurable magnitudes didn't prevent them in answering satisfactorily the other questions about the real numbers. However our didactical/pedagogical experience suggests that the teaching of geometric representations of real numbers at school helps in general their better understanding by students. We shall close this section with some comments on it.

It seems that within the culture of ancient Greek mathematics the geometric figure was the basis for unfolding mathematical thought, since it helped in obtaining conjectures, fertile mathematical ideas and justifications (proofs). In fact, convincing arguments are built by drawing auxiliary lines, optical reformations and new modified figures, and therefore mathematical thinking becomes more completed in this way. For example, the invention of the existence of incommensurable line segments by the Pythagorean philosophers was the starting point for the discovery of irrational numbers.

Most of irrational numbers, like $\sqrt[3]{2}$, π, e, etc., correspond to lengths of line segments that cannot be constructed by ruler and compass only. Nevertheless, at school level we correspond all these numbers to points of the real axis in an axiomatic (or approximate, if you prefer to call it so), way, which usually is not clearly understood by students (actually it is based on the principle of nested intervals). At the 27th Panhellenic Conference on Mathematics Education of the Greek Mathematical Society that took place in Chalkida (2010) we had the opportunity to hear the description of an experienced colleague, who is teaching for years in a very good private school (Gymnasium) and who became embarrassed when she was asked by a student the following question: "*Are there any circles whose length of circumference is a rational number? For example does it happen for the circle of radius 1/π?*" Algebraically speaking the student's remark was logical. The problem however is that the length 1/π and therefore the corresponding cycle also cannot be geometrically constructed!

In contrast to the ancient Greek mathematics, numerical thought is the most frequently used at school today. This is logically expected, since numerical excels geometrical culture in our contemporary world and therefore it plays the main role in representations that students build at school. Nevertheless, we have the feeling that the excessive use of numerical arguments wounds the geometrical intuition. In fact, we believe in general that a rich experience of students with geometric forms, before being introduced to numerical arguments and analytical proofs, is not only useful, but it is indispensable (Arcavi et al., 1987). The geometric representations of real numbers enrich their teaching, connecting it historically with the discovery of existence of incommensurable magnitudes and the relevant theory of Eudoxus. Activities of geometric constructions of irrational numbers could be organized in classroom combining history of mathematics with Euclidean Geometry, like the problem of doubling the volume of a cube (Delion problem), which is appeared in Plato's dialogue "Menon" (Kosyvas & Baralis, 2010).

Based on those discussed in the present section, we conclude that, since the probability of appearance of opaque representations of rational and irrational numbers is high from one hand, and because of the existence of transcendental numbers on the other hand, some voids, inconsistencies, or misconceptions remain often to students, but even to adults after finishing school, concerning the understanding of real numbers. Therefore teacher's attention is necessary in preventing such phenomena.

We finally ought to clarify that all that we have discussed here are simply some ideas aiming to help the instructor towards the difficult indeed subject of the didactic approach of real numbers at school level. However, by no means they could be considered as an effort to introduce, or even more to impose, a model of teaching, because our belief is that the effort of introducing such a model is actually a utopia! In fact, teacher should be able to make a small "local research", readapting methods and plans of the teaching process according to the special conditions of his (her) class (Voskoglou, 2009).

In general lines our didactic proposition includes: A fertile utilization of already existing informal knowledge and beliefs about numbers, active learning through rediscovery of concepts and conclusions, construction of knowledge by students individually, or as a team, in classroom. Construction of knowledge follows in general student's optical corner, while teacher's role is limited to the discussion in the whole class of wrong arguments and misinterpretations observed. The teaching process could be based on multiple representations of real numbers (rational numbers written as fractions and periodic decimals, irrational numbers considered as non rational ones and as incommensurable decimals, which are limits of sequences of rational numbers, roots, geometric representations, etc.) and on flexible transformations among them.

Conclusions

The understanding of irrational numbers is fundamental for students of secondary education in reestablishing and extending the notion of numbers. Nevertheless, the transition from the set of rational numbers to the set of real numbers strikes against inherent difficulties, connected to the incomplete understanding of rational number and to the nature of irrational numbers.

According to the mathematics curricula of secondary education and the restricted abilities of students' at this age in understanding abstract and difficult concepts, the only suitable method for presenting the real numbers at school is by using their decimal representations.

Our basic hypothesis for our experimental research reported in this article was that the main obstacle for the understanding of real numbers is the difficulties that students face in dealing with their multiple *semiotic representations*, i.e. the ways in which we describe and we write them down.

The first part of our research was based on students' written response to a properly designed questionnaire. The novelty of this study has to do with the choice of the subjects of our experiment, consisting of high-school students (13-14 years old) a few months after learning about the real numbers and students of a graduate technological institute (18-19 years old) using mathematics as a tool for studying and better understanding their sciences (prospective engineers and economists). As far as we know a similar experiment was performed in the past only by Fischbein et al. (1995) with high-school students and prospective teachers of mathematics, while analogous experiments performed by other researchers with prospective or pre-service teachers only.

The quantitative results of our experiment showed an almost complete failure of the technologist students to deal with processes connected to geometric constructions of incommensurable magnitudes. However, and contrary to our hypothesis about the role of their semiotic representations for the understanding of real numbers, this didn't prevent them in answering satisfactorily the other questions. In fact, although the majority of them correspond to mediocre graduates of the secondary education, the superiority of their correct answers with respect to those of high-school students was evident in most cases. This is a strong indication that the age and the width of mathematical knowledge of the individual play an important role for the better understanding of the real numbers. This is crossed by the findings of Fischbein et al. (1995), which however were more or less expected, since they concern prospective teachers of mathematics.

In general (with the exception of the geometric representations) our basic hypothesis was verified by the experiment's results, since students' performance was connected to their ability of flexible transformations among the multiple representations of real numbers. Apart from the above contributions to the research literature, the results of our experiment verified also findings of experiments performed by other researchers, connecting students' difficulties in understanding the real numbers with the incomplete understanding of rational numbers, the incommensurability and nondenumerability of irrational numbers, the frequently appeared opaque representations of rational and irrational numbers, etc.

Reflecting on certain characteristic answers appeared in the completed questionnaires of our experimental research we considered useful to penetrate deeper to the reasons that urged students in giving these answers. Therefore we decided to make a complementary qualitative research by taking some interviews from students. From the dialogues of these interviews presented above it becomes (among the others) evident again the students' *difficulty to deal successfully with the multiple representations of real numbers*. A theoretical explanation about this was obtained through the adoption of the conceptual framework of dimensions of knowledge, introduced by Tirosh et al. (1998) for studying the comprehension of rational numbers.

Following in part the idea of generic decomposition of the APOS analysis (Weller et al., 2009) we suggested a possible order for development of understanding the real numbers by students when teaching them at school. Based on those discussed we concluded that, since the probability of appearance of opaque representations of rational and irrational numbers is high from the one hand, and because of the existence of transcendental numbers on the other hand, some voids, inconsistencies, or misconceptions remain often to students, but even to adults after finishing school, concerning the understanding of real numbers. Therefore teacher's attention is necessary in preventing such phenomena.

In general terms, our didactic proposition includes a fertile utilization of the already existing informal knowledge and beliefs about numbers, active learning through rediscovery of concepts and conclusions, construction of knowledge by students individually, or as a team, in classroom. The teaching process could be based on multiple representations of real numbers and on flexible transformations among them.

Open Questions - Epilogue

The discussion made in this article marked out the following open to further study and research questions concerning the understating and teaching of real numbers at school:

- How useful is for their better understanding the enrichment of teaching of real numbers with geometric representations? The data of our classroom experiment did not permit us to obtain an explicit conclusion about this, since the almost total failure of T.E.I. students' in constructing geometrically incommensurable lengths and/or in corresponding them to points of the real axis did not seem to prevent them in answering successfully the other questions.
- How students could understand better the approximate/ axiomatic correspondence of incommensurable magnitudes that cannot be geometrically constructed to points of the real axis? For example, for the construction of length $\sqrt[3]{2}$ (doubling the volume of a cube with edge equal to the unit of lengths) we could use the graph of function $f(x) = \sqrt[3]{x}$ (or $f(x) = \sqrt[3]{x - 2}$) constructed in absolute exactness (Sirotic & Zazkis, 2007b). Nevertheless this could be succeeded only by the help of a computer, which means that it will be a distance between the theoretic and the practical approach of the problem.
- Which is the proper way, for each level of education, to study the continuum of R in contrast to the density of Q? In other words how students could be persuaded that in a given interval (of numbers, or of points, if we consider the real axis) it is possible to have an infinity of elements of a certain

> type (rational numbers, or points) when this is not compatible with usual logic and our intuition?
> • How we could communicate to students the image of mathematics as an organized whole, where the systems of numbers play an important role? In this way students could get the feeling of the grandeur, the beauty of mathematics as a fundamental human achievement, not only its utility for practical matters (Fischbein et al. 1995).

In answering last question it is heard faintly to suggest a turn to "new mathematics", where the whole teaching is based on theory of sets, algebraic structures and mathematical logic, like it happened with educational reform of 1960's, that was proved to be a complete failure. Nevertheless, it could be useful to be examined, if and how much the teaching, in a simple and practical approach, of some elements from theory of algebraic structures at the last class of the secondary education, could help for a better and deeper understanding of real numbers. More explicitly, that *Q* and *R* with respect to the known properties of addition and multiplication (subtraction is defined in terms of addition, and division in terms of multiplication) have the structure of a *field* (it is not necessary to give the definitions of a group and a ring before and the corresponding axiomatic foundations), and the concept of *isomorphism* as a 1-1 correspondence between fields "preserving" the properties of operations. For example, the concept of isomorphism could help students to understand why the set of all series $\sum_{n=0}^{\infty} k^n/10^n$, with $k_0 \in Z$, and $k_1, k_2, \ldots k_v, \ldots$ natural numbers less than 10, not all equal to 9, coincides in practice with the set of real numbers (see Appendix 2), and, later on, why the same happens with R^2 and the field *C* of complex numbers. All these could be taught either in parallel with reviewing the basic sets of numbers, that usually precedes the teaching of complex numbers, or as part of a voluntary, experimental course, together with other mathematical regularities.

We are aware that the above idea will possibly give rise to critiques of the form: "When constructivism is today the predominant theory for learning, such formalistic approaches are out of place and time". Nevertheless our belief is that in matters like this we must not be absolute. In fact, none of epistemological/philosophical trends in mathematics and its didactics could be considered as the perfect one. Each one of them has its advantages and its weak points that affect in an analogous way the march of mathematical science. Therefore the required thing is to find a kind of "balance" among them (Voskoglou 2007, section 5), so that to be able to drive forward more effectively a combined scientific and didactic vision for research and teaching of mathematics.

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Appendix 1

List of questions of our experimental research

1. Which of the following numbers are natural, integers, rational, irrational and real numbers?

$$-2,\quad -\frac{5}{3},\quad 0,\quad 9{,}08\ ,\quad 5\ ,\quad 7{,}333\ldots,\quad \pi = 3{,}14159\ldots,\quad \sqrt{3}\ ,\quad -\sqrt{4}\ ,\quad \frac{22}{11},$$

$$5\sqrt{3},\quad -\frac{\sqrt{5}}{\sqrt{20}}\ ,\quad \left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right),\quad -\frac{\sqrt{5}}{2},\quad \sqrt{7}-2,\quad \sqrt{\left(\frac{5}{3}\right)^2}$$

(Units 2)
2. Are the following inequalities correct, or wrong? Justify your answers.

$\frac{2}{3} < \frac{14}{21}$, $\frac{2002}{1001} > 2$ (Unit 1)

3. Which is the exact quotient of the division 5÷7? (Unit 1)
4. Convert the fraction 7/3 to a decimal number. What kind of decimal number is this and why we call it so? (Unit 1)
5. Are 2.8254131131131… and 2.0013131131113111… periodic decimal numbers? In positive case, which is the period? (Units 1,5)
6. Find the square roots of 9, 100 and 169 and describe your method of calculation. (Unit 1)
7. Find the integers and the decimals with one decimal digit between which lies √2. Justify your answers. (Units 1,5)
8. Characterize the following expressions by C if they are correct and by W if they are wrong: √2 = 1.41, √2 =1.414444…, √2 ≈ 1.41, there is no exact price for √2. (Units 1,5)
9. Find two rational and two irrational numbers between √10 and √20. How many rational numbers are there between these two square roots? (Unit 1)
10. Find two rational and two irrational numbers between 10 and 20. How many irrational numbers are there between these two integers? (Unit 1)
11. Are there any rational numbers between 1/11 and 1/10? In positive case, write down one of them. How many rational numbers are between the above two fractions? (Unit 1)
12. Are there any rational numbers between 10.20 and 10.21? In positive case, write down one of them. How many rational numbers are in total between the above two decimals?
13. Characterize the following expressions as correct or wrong. In case of wrong ones write the corresponding correct answer.

$\sqrt{3+5} = \sqrt{3} + \sqrt{5}$, $\sqrt{3 \cdot 7} = \sqrt{3} \cdot \sqrt{7}$, $\sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$, the unique solution of the equation $x^2 = 3$ is $x = \sqrt{3}$, $\sqrt{\left(1-\sqrt{17}\right)^2} = 1 - \sqrt{17}$. (Units 2)

14. Construct, by making use of ruler and compass only, the line segments of length $\sqrt{2}$ and $\sqrt{3}$ respectively and find the points of the real axis corresponding to the real numbers $\sqrt{2}$ and $-\sqrt{3}$. Consider a length of your choice as the unit of lengths. (Units 2,5)

15. Is it possible for the sum of two irrational numbers to be a rational number? In positive case give an example. (Unit 1)

Appendix 2

Discussion on the decimal representations of the real numbers and the equation 0.999... =1

In most books on Number Theory and Number Systems (e.g. Hardy & Wright 1993, Sierpinski 1988, Feferman 1989, etc) it is argued that a non negative real number, say x, is expressed as a decimal, or equivalently it has a decimal representation, if

$$x=[x]+\frac{c_1}{10}+\frac{c_2}{10^2}+\frac{c_3}{10^3}+\ldots\ldots \qquad (1)$$

In the above expression [x] denotes the integral part of x (i.e. the largest integer not exceeding x) and c_i, i=1,2,3,…, are integers such that $0 \le c_i \le 9$. We write then $x=[x],c_1c_2c_3\ldots$ A negative real number can be expressed as a decimal by using the decimal expansion of its opposite number in the obvious way.

It is well known that any non negative real number x has a decimal representation of the form (1) (e. g. Kalapodi 2010; Theorem 3.2). More specifically, if x has a finite decimal representation, then it has exactly two decimal representations (Kalapodi 2010; Theorem 3.7); e.g. 2.5 = 2.5000… = 2.4999… On the other hand, if *x* has no finite decimal representation (infinite decimal), then it has a unique decimal representation, in which there exist infinitely many c_i’s different from 9 (Kalapodi, Theorem 3.5 and Theorem 4.5). We recall that a decimal representation of the form (1) is called finite, if there exists an index i_0 such that $c_i = 0$, for all $i \ge i_0$. Notice that, in any decimal representation of the form (1) at least one of the c_i’s must be different from 9. In fact, assume that $x = [x] + \sum_{i=1}^{\infty} 9/10^i$ (2) is a decimal representation of the form

(1). Then, since $\sum_{i=1}^{\infty} 9/10^i$ is a decreasing geometric series with common ration 1/10, we get that $\sum_{i=1}^{\infty} 9/10^i = (9/10) / (1-(1/10)) = 1$. Thus x=[x] +1, which is impossible, since, according to its definition, [x] is the largest integer not exceeding x. Consequently, all the expressions of the form (2) cannot be accepted as decimal representations of real numbers in the sense of definition (1). In particular, although the series $\sum_{i=1}^{\infty} 9/10^i$ converges to 1, we can not accept the form 0.999... as a decimal representation of 1. The question arising under the above data is what is actually the meaning of the symbol $\kappa_0.999\ldots$, with κ_0 a non negative integer. Having in mind that instead of saying that the sum (i.e. the limit of the sequence of its partial finite sums) of a given series, say Σ, is equal to α, we usually write Σ=α, where the symbol "=" has not the usual meaning of equality in this case, the answer could be that the above symbol represents the series $\kappa_0 + \sum_{n=1}^{\infty} 9/10^n$ and not its sum, which is equal to the real number κ_0+1. A number of colleagues believe that, for reasons of mathematical consequence, we must accept in general that all symbols of the form $\kappa_0,\kappa_1\kappa_2\ldots\ \kappa_n\ldots$, with κ_0 a nonnegative integer and $\kappa_1, \kappa_2,\ldots, \kappa_n,\ldots$ natural numbers less than 10, represent the series $\sum_{n=1}^{\infty} \kappa_n/10^n$ and not its sum, which is equal to the corresponding real number. Consequently the representation of real numbers as infinite decimals has no meaning at all!

Fortunately the results obtained when using these representations are conventionally correct, because the corresponding operations could be performed in an analogous way among the sequences of the partial sums of the corresponding series. This allows us to pass through this sensitive matter at school level without touching it at all. However, from the above analysis it becomes evident that all the above problems (let me characterize them as pseudo problems, because, as we'll see below, they can easily be solved) are created due to the fact that the definition of the decimal representations of real numbers is given in the form (1). In fact, one can extend definition (1) by accepting that any positive integer, say k, apart from its usual (let us call it main) decimal representation, has also another one (let us call it secondary) of the form x = k-1.999…, where [x]= k. In particular the secondary decimal representation of 1 is x=0.999…, with [x] = 1.

Probably, an easy way to avoid giving all these explanations at school level (which obviously could create confusion to students) is to define the set *R* of real numbers in terms of their decimal representations. In fact, from the above analysis it becomes evident that in order to consider each real number only once, one must take into account only the decimal expressions (and their opposite numbers) of the form $a,c_1c_2c_3\ldots$, with a and c_i (i=1,2, 3,…) integers, $a \geq 0$, $0 \leq c_i \leq 9$, where there exists an index i_0 such that it is not $c_i = 9$ for all $i \geq i_0$.

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